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THE THERMALLY STRESSED STATE OF AN ELASTIC HALF-PLANE HEATED BY A UNIFORMLY MOVING HEAT SOURCE†

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A solution of the plane problem of the motion of a heat source with constant velocity along the boundary of an elastic half-plane is constructed in a development of the method proposed previously in [1] for finding fundamental thermoelastic solutions in the case of problems of this type. It is assumed that the boundary of the half-plane is stress-free and that heat exchange with the surrounding medium occurs in accordance with Newton's law. It is further assumed that the source velocity of motion is small, by virtue of which inertial effects in the half-plane are ignored. The assumption is also made that the physicomechanical properties of the half-plane are independent of the temperature and that the effect of thermoelastic connectivity can be neglected. A Fourier integral transform, the inversion of which is performed by contour integration methods, is used to solve the problems of heat conduction and thermoelasticity in question. As a result, formulae are obtained for the temperature of the half-plane and the stresses and strains in it. Results of calculations are presented.

In order to find the thermally stressed state which arises as a consequence of the frictional generation of heat at a sliding contact, it is necessary to determine the temperature, the stresses and the displacements in the half-plane for a heat source which moves along its surface. There are several approaches to the determination of these quantities. Serious difficulties of a computational nature arise when summing slowly converging Fourier series when realizing the concept of a sinusoidal temperature wave which moves uniformly over the surface of an elastic half-plane in combination with the use of thermoelastic displacement potential [2, 3]. Analytic expressions for the quasisteady surface displacements and shear stresses which are applicable for arbitrary values of the Peclet number have been obtained in [5] by solving a heat conduction problem in the case of an instantaneous heat source which acts on the surface of an elastic half-plane [4].

An asymptotic solution has been constructed for large (>10) values of the Peclet parameter which determines the heat flux distribution in each of the bodies in contact. A finite-element method was employed in [7] for this purpose.

It is assumed in all of the papers mentioned above that the surface of the half-plane, outside the region which is heated, is thermally isolated. The solution of the quasi-steady heat conduction problem for a heat source which moves uniformly over the surface and which takes account of heat exchange with the external medium in accordance with Newton's law was obtained in [1]. The corresponding thermal stresses and strains were determined later in [8] for large values of the Peclet number.

1. THE HEAT CONDUCTION PROBLEM

The problem is formulated within the framework of the classical linear theory of thermoelasticity. The heat conduction equation in a system of xy coordinates is rigidly connected with the source which moves at a constant velocity V along the boundary $y = 0$ of the half-plane has the form

$$\Delta T + \beta T_x = 0, \ |x| < \infty, \ y \geqslant 0, \ \beta = V/k \tag{1.1}$$

(k is the thermal diffusivity). We shall construct a solution of Eq. (1.1) which satisfies the boundary conditions

$$KT_y - hT = -Q\delta(x), \ |x| < \infty, \ y = 0$$

$$T, T_x, T_y \to 0, \ r \equiv (x^2 + y^2)^{1/2} \to \infty$$

(K is the thermal conductivity) using a Fourier integral transform with respect to the variable x. We will represent the resulting solution in the form

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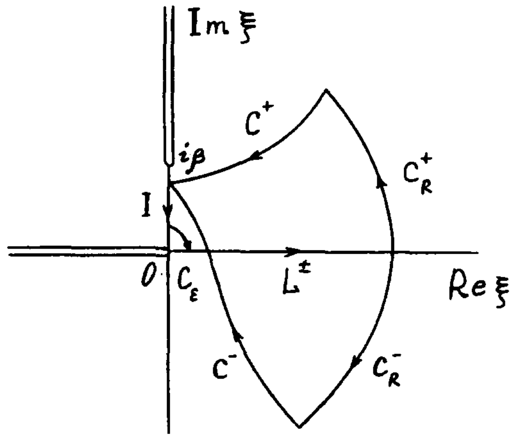

Fig. 1.

$$T(x,y)=\frac{Q}{\pi K}\operatorname{Re}\left\{\int_0^\infty \exp[-y\eta(\xi)]P(x,0,\xi)d\xi\right\} \tag{1.2}$$

$$P(x,y,\xi)=\exp[-\xi(y-ix)]/[\gamma+\eta(\xi)],\quad \gamma=h/K,\quad \eta(\xi)\equiv\sqrt{\xi^2-i\beta\xi}$$

As a result of integration along the closed contours (Fig. 1)

$$\Gamma^\pm=L^\pm\cup C_R^\pm\cup C^\pm\cup I\cup C_\varepsilon \tag{1.3}$$

(the plus and minus superscripts denote integration along the curve Γ^+ when $x>0$ and Γ^- when $x<0$), we find the temperature at an arbitrary point of the half-plane from relation (1.2)

$$T(x,y)=\frac{Q}{\pi K}\exp(-s_0)\int_0^\infty\frac{R(x,y,s)e^{-s}ds}{\sqrt{s^2+(2s_0-\beta x)s}} \tag{1.4}$$

$$R(x,y,s)=\frac{s^2+a_1s+b_1}{s^2+a_2s+b_2},\quad s_0=\frac{1}{2}\beta\left(x+\sqrt{x^2+y^2}\right)$$

$$a_1=\beta r+\gamma y,\quad a_2=\beta r+2\gamma y,\quad b_1=(\beta y/2)^2+\gamma\beta yr/2,\quad b_2=(\beta y/2)^2+\gamma\beta yr+\gamma^2r^2$$

We now consider two well-known special cases of the solution (1.4)

1. $h=0, y\geqslant 0$; then, $\gamma=0$, $R(x,y,s)=1$ and

$$T(x,y)=\frac{Q}{\pi K}\exp\left(-\frac{\beta x}{2}\right)K_0\left(\frac{\beta r}{2}\right) \tag{1.5}$$

($K_0(\cdot)$ is a modified Bessel function of the second kind);

2. $h\neq 0, y=0$; then, the temperature of the boundary points of the convectively cooled half-plane is equal to

$$T(x,0)=\frac{Q}{\pi K}\exp(-s_0)\int_0^\infty\frac{\sqrt{s^2+s}\,\exp(-\beta|x|s)ds}{s^2+s+\alpha^2},\quad \alpha=\frac{\gamma}{\beta} \tag{1.6}$$

Relations (1.5) and (1.6) were obtained for the first time in [4] and [1] respectively.

2. THE PROBLEM OF THERMOELASTICITY

We shall represent the components of the thermal stresses in terms of the Airy function Φ and the thermoelastic displacement potential Ψ in the form [9]

$$\sigma_x=F_{,yy},\quad \sigma_y=F_{,xx},\quad \sigma_{xy}=-F_{,xy},\quad F\equiv\Phi-2\mu\Psi \tag{2.1}$$

(μ is the shear modulus).

We determine the functions Φ and Ψ from the solution of the thermoelastic boundary-value problem

$$\Delta\Delta\Phi = 0,\ \Delta\Psi = NT,\ |x| < \infty,\ y \geqslant 0;\ N = (1+\nu)\alpha_T/(1-\nu) \tag{2.2}$$

$$\sigma_y(x, 0) = \sigma_{xy}(x, 0) = 0,\ |x| < \infty \tag{2.3}$$

$$\sigma_x, \sigma_y, \sigma_{xy} \to 0 \text{ as } r \to \infty \tag{2.4}$$

Here, ν is Poisson's ratio and α_T is the coefficient of linear thermal expansion.

We find a solution of the boundary-value problem (2.2)–(2.4) using a Fourier integral transform with respect to the variable x. We have

$$F(x, y) = 2\mu N\beta^{-1}[L(x, y) + M(x, y)] \tag{2.5}$$

Here

$$L = L_1 + y(L_2 + L_3) \tag{2.6}$$

$$L_1(x,y) \equiv \frac{Q}{\pi K}\operatorname{Re} L_1^*(x,y) = \frac{Q}{\pi K}\operatorname{Re}\left\{i\int_0^\infty \frac{P(x,y,\xi)d\xi}{\xi}\right\} \tag{2.7}$$

$$L_2(x,y) \equiv \frac{Q}{\pi K}\operatorname{Re} L_2^*(x,y) = \frac{Q}{\pi K}\operatorname{Re}\left\{-i\int_0^\infty \frac{\eta(\xi)P(x,y,\xi)d\xi}{\xi}\right\} \tag{2.8}$$

$$L_3(x,y) \equiv \frac{Q}{\pi K}\operatorname{Re} L_3^*(x,y) = \frac{Q}{\pi K}\operatorname{Re}\left\{i\int_0^\infty P(x,y,\xi)d\xi\right\} \tag{2.8}$$

$$M(x,y) \equiv \frac{Q}{\pi K}\operatorname{Re} M^*(x,y) = \frac{Q}{\pi K}\operatorname{Re}\left\{-i\int_0^\infty \frac{\exp[-y\eta(\xi)]P(x,0,\xi)d\xi}{\xi}\right\} \tag{2.9}$$

It follows from formula (2.5) that the stress function F is the sum of two integrals. The first of these L (2.6) corresponds to the biharmonic function Φ and has the form of the transform of the Laplace integral transformation of a certain function when $y \geqslant 0$ with respect to the variable ξ. The second integral, M (2.9) is related to the thermoelastic stress potential Ψ

$$\Psi(x, y) = 2\mu N\beta^{-1}M(x, y) \tag{2.10}$$

where

$$M_{,x} = T,\ M_{,yy} = -T_x - \beta T \tag{2.11}$$

We shall use the notation

$$\begin{gathered} y(L_{2,xx}^* + L_{3,xx}^*) \equiv S_1,\ iL_{3,x}^* = L_{3,y}^* = L_{1,xx}^* = -L_{1,xy}^* = -iL_{1,yy}^* \equiv S_2 \\ L_{2,x}^* = -iL_{2,y}^* \equiv S_3 \end{gathered} \tag{2.12}$$

Differentiating the stress function F (2.5), in accordance with formulae (2.1), when account is taken of (2.10)–(2.12), we find

$$\begin{gathered} \sigma_x = N_1\operatorname{Re}(-S_1 + S_2 + 2iS_3 - \beta T^* - T_{,x}^*),\ \sigma_y = N_1\operatorname{Re}(S_1 + S_2 + T_{,x}^*) \\ \sigma_{xy} = -N_1\operatorname{Re}(iS_1 + S_3 + T_{,y}^*),\ N_1 = 2\mu NQ(\pi K\beta)^{-1},\ T^* = \pi KT/Q \end{gathered} \tag{2.13}$$

We note that, when $y = 0$, it follows from formulae (2.13) that $\sigma_y = \sigma_{xy} = 0$.

The elastic displacements are related to the thermal stresses using the Duhamel–Neumann formulae [9]

$$\begin{gathered} 2\mu u_{,x} = (1-\nu)\sigma_x - \nu\sigma_y + 2\mu(1+\nu)\alpha_T T \\ 2\mu \upsilon_{,y} = (1-\nu)\sigma_y - \nu\sigma_x + 2\mu(1+\nu)\alpha_T T \end{gathered} \tag{2.14}$$

Substituting the value of the stresses (2.13) into the right-hand sides of relations (2.14) and integrating with respect to x and y, we obtain

$$u = N_2 \operatorname{Re}\{(1-\nu)i[2L_2^* + L_3^* + yL_{23}^*] - \nu L_{,x}^* - T^*\} \tag{2.15}$$

$$\upsilon = N_2 \operatorname{Re}\{(1-\nu)[L_2^* + 2L_3^* - yL_{23}^*] - \nu L_{,y}^* - M_{,y}^*\}$$

$$N_2 = N_1/(2\mu),\quad L_{23}^* = S_2 + iS_3,\quad L_{,x}^* = iS_3 - iyL_{23}^*,\quad L_{,y}^* = L_2^* + yL_{23}^*$$

3. FINDING THE FUNCTIONS L_2^*, L_3^* AND $M_{,y}^*$

It follows from relations (2.13) and (2.15) that the components of the thermal stress tensor and the displacement vector can be expressed in terms of the functions L_2^*, L_3^*, T^* their derivatives and, also, the function $M_{,y}^*$. The temperature at an arbitrary point of the half-space is given by formula (1.4).

We now consider the integrals L_j^*, $j = 2, 3$ (2.7), (2.8) and use the notation

$$s \equiv \xi(y - ix),\quad 2s_\pm^2 \equiv \pm(s^2 - \beta xs) + \sqrt{(s^2 - \beta xs)^2 + \beta^2 y^2}$$

Then

$$\sqrt{\xi^2 - i\beta\xi} = (S_+ - iS_-)/(y - ix)$$

and the functions L_2^*, L_3^*, take the form of the transforms of Laplace integral transformations with the transformation parameter $p = 1$.

$$L_j^*(x,y) = \int_0^\infty \bar{L}_j(x,y,s)e^{-s}ds,\quad \bar{L}_j = \frac{A_j}{D},\quad j = 2,3 \tag{3.1}$$

$$sA_2 = \gamma(xs_+ - ys_-) - i[s_+(s_+ + \gamma y) + s_-(s_- + \gamma x)]$$

$$A_3 = -(s_- + \gamma x) + i(s_+ + \gamma y),\quad D = (s_+ + \gamma y)^2 + (s_- + \gamma x)^2$$

When account is taken of formula (2.9), the integral

$$M_{,y}^* = i\int_0^\infty \frac{\eta(\xi)\exp[-y\eta(\xi)]P(x,0,\xi)d\xi}{\xi}$$

reduces to a form which is convenient for computerized calculation using integration along closed contours (1.3). As a result, we obtain

$$M_{,y}^*(x,y) = \int_{S_0}^\infty \bar{M}_{,y}(x,y,s)e^{-s}ds \tag{3.2}$$

$$\bar{M}_{,y}(x,y,s) = \frac{\left(-\gamma\xi_- + i(\xi_+^2 + \xi_-^2 + \gamma\xi_+)\right)(B - i\beta y)}{s\left((\gamma + \xi_+^2) + \xi_-^2\right)B},\quad B = \sqrt{4s^2 - 4\beta xs - \beta^2 y^2}$$

$$\xi_+ = r^{-2}y(2s - \beta x)/2,\quad \xi_- = r^{-2}xB/2,\quad \sqrt{\xi^2 - i\beta\xi} = \xi_+ + i\xi_-$$

If the surface of the half-plane is thermally insulated ($h = 0, y \neq 0$), it then follows from formulae (3.1) that

$$\bar{L}_2 = -is^{-1},\quad \bar{L}_3 = i[s^2 - i\beta(y - ix)s]^{-1/2}$$

From where [10], we obtain

$$L_2^* = i\ln|_{y-ix}^\infty,\quad L_3^* = i\exp[-i\beta(y - ix)/2]K_0[-i\beta(y - ix)/2] \tag{3.3}$$

We note that the function is differentiable although it is unbounded. When account is taken of the relation [11]

$$K_0(z) = \pi i H_0^{(1)}(iz)/2,\quad -\pi < \arg(z) < 0$$

($H_0^{(1)}(\cdot)$ is the Hankel function), we find from (3.3) that

$$L_3^* = -\pi\exp[-i\beta(y-ix)/2]H_0^{(1)}[\beta(y-ix)/2]/2$$

On the basis of (3.2), the real part of the function M_y^* when $\gamma = 0, y \neq 0$ has the form

$$\operatorname{Re} M_y^* = -\frac{\pi}{2} + \arccos\left(\frac{x}{r}\right) - \frac{\beta y}{2}\int_0^1 \exp\left(-\frac{\beta xs}{2}\right)K_0\left(\frac{\beta rs}{2}\right)ds$$

If, however, $y \neq 0, y = 0$, then, from relation (3.1), we find

$$L_2^* = -i\left[\alpha\int_0^1 s^{-1}\sqrt{s-s^2}\,E_-(x,s)ds + \int_0^\infty (s-1)E_-(x,s)ds + \right.$$

$$\left. + \alpha\exp(-\beta x)\int_0^\infty (s^2+s)^{-1/2}E_+(x,s)ds, \quad x>0 \right. \tag{3.4}$$

$$L_2^* = -i\int_0^\infty (s-1)E_-(x,s)ds - \alpha\int_0^\infty (s+1)(s^2+s)^{-1/2}E_+(x,s)ds, \quad x<0$$

$$E_\pm(x,s) = \exp(-\beta|x|s)/(s^2 \pm s + \alpha^2)$$

In the case of the function L_3^*, the corresponding formulae take the form

$$L_3^* = -\int_0^1 (\sqrt{s^2+s}+\alpha)^{-1}\exp(-\beta xs)ds - \alpha\exp(-\beta x)\int_0^\infty E_+(x,s)ds +$$

$$+ i\exp(-\beta x)\int_0^\infty \sqrt{s^2+s}\,E_+(x,s)ds, \quad x>0 \tag{3.5}$$

$$L_3^* = i\int_0^1 \sqrt{s^2+s}\,E_+(x,s)ds + \alpha\int_0^\infty E_+(x,s)ds, \quad x<0$$

The real part of the function M_y^* is equal to $-\operatorname{Re} L_2^*$ in the case under consideration.

4. ASYMPTOTIC BEHAVIOUR

For the solution of contact problems of thermoelasticity taking the frictional generation of heat into account it is necessary to have expressions for the normal thermal displacements of the surface of the bodies in contact. When $y = 0$, we have from the second of relations (2.15) that

$$\upsilon(x,0) = 2(1-\nu)N_2\operatorname{Re}(L_2^* + L_3^*)$$

from which, when account is taken of formulae (3.4) and (3.5), we obtain

$$\upsilon(x,0) = -2(1-\nu)N_2V(x) \tag{4.1}$$

$$V(x) = H(x)\int_0^1 \frac{\exp(-\beta xs)ds}{\sqrt{s-s^2}+\alpha} - \alpha\begin{cases}\exp(-\beta x)\ V_+(x), & x>0\\ V_-(x), & x<0\end{cases} \tag{4.2}$$

$$V_+(x) = \int_0^\infty (s-\sqrt{s^2+s})(s^2+s)^{-1/2}E_+(x,s)ds$$

$$V_-(x) = \int_0^\infty (\sqrt{s^2+s}-s-1)(s^2+s)^{-1/2}E_+(x,s)ds \tag{4.3}$$

($H(\cdot)$ is the Heaviside function).

The integrands in expressions (4.3) are rapidly decreasing functions as $s \to \infty$. Hence, the principal contribution to $V_\pm(x)$ will be determined by the behaviour of these functions in the neighbourhood of zero. Assuming that $0 \leqslant s \leqslant \delta \ll 1$ and $\alpha < \sqrt{\delta}$, we find

$$V_+(x) \cong J_2(x,\delta) - J_1(x,\delta), \quad V_-(x) \cong J_1(x,\delta) - J_2(x,\delta) - J_3(x,\delta) \tag{4.4}$$

Here

$$J_1(x,\delta) = \ln|\alpha^2+\delta| - \ln|\alpha^2| + \beta|x|\,(\delta - \alpha^2\ln|\alpha^2+\delta| + \alpha^2\ln|\alpha^2|)$$

$$J_2(x,\delta)=2[\delta-\alpha\,\mathrm{arctg}(\delta^{1/2}/\alpha)]+2\beta|x|\,[\delta^{3/2}/3-\alpha^2\delta^{1/2}+\alpha^3\,\mathrm{arctg}(\delta^{1/2}/\alpha)] \quad (4.5)$$

$$J_3(x,\delta)=2\,\mathrm{arctg}(\delta^{1/2}/\alpha)/\alpha+2\beta|x|\,[\delta-\alpha\,\mathrm{arctg}(\delta^{1/2}/\alpha)]$$

When $\alpha \to 0$, it follows from relations (4.4) and (4.5) and $V_+(x) = 0$, $V_-(x) = -\pi/\alpha$. Since [12]

$$\int_0^1 \frac{\exp(-\beta|x|s)ds}{\sqrt{s-s^2}} = \pi\exp\left(-\frac{\beta x}{2}\right)I_0\left(\frac{\beta|x|}{2}\right)$$

we have

$$V(x)=\begin{cases}\pi\exp(-\beta x/2)I_0(\beta|x|/2), & x>0\\ \pi, & x<0\end{cases} \quad (4.6)$$

($J_0(\cdot)$ is the modified Bessel function of the first kind).

Relation (4.1), in the case of $V(x)$ (4.6), is identical to the well-known result in [5].

5. NUMERICAL ANALYSIS

The distribution of the dimensionless temperature $T^*(x, 0)$ of the surface of the elastic half-plane, calculated using formula (1.6) with $\beta = 1$ and different values of γ, is shown in Fig. 2. The results for $\gamma = 0.01$ are identical with an accuracy of 10^{-4} to the data obtained using formula (1.5) in the case when the surface $y = 0$ of the half-plane is thermally insulated.

The change in the normal stress $V(x)$ of the boundary of the half-plane, found when $\beta = 1$ for different α using formula (4.2), is shown in Fig. 3. The results represented by the dashed line were obtained by means of calculations

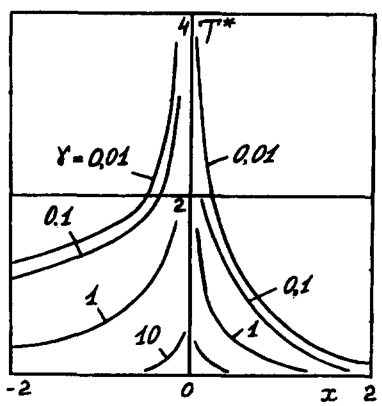

Fig. 2.

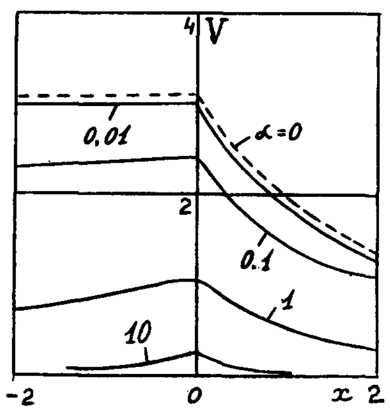

Fig. 3.

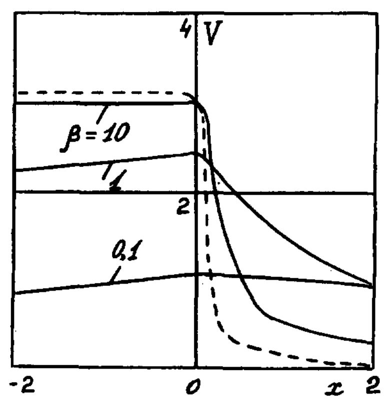

Fig. 4.

using formula (4.6). At small (of the order of 0.01 or less) values of the parameter α, the asymptotic relations (4.4) and (4.5) can be used to calculate the normal displacements.

The change in $V(x)$ for a fixed value of the parameter $\gamma = 0.1$ and various β is shown in Fig. 4. Data from a calculation using formula (4.6) with $\beta = 10$ are represented by the dashed line. An analysis of the results of the calculations showed that formula (4.6) can be used when $\gamma < 0.1$. In this case, when the parameter β is increased (greater than 10), the behaviour of the displacement $V(x)$ can be approximated to a high degree of accuracy by the step function $H(-x)$.

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